A Hybrid Method of Combinatorial Search and Gradient Descent for Discrete Optimization

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Abstract

Discrete optimization is a central problem in mathematical optimization with a broad range of applications, among which binary optimization and sparse optimization are two common ones. However, these problems are NP-hard and thus difficult to solve in general. Combinatorial search methods such as branch-and-bound and exhaustive search find the global optimal solution but are confined to small-sized problems, while gradient descent methods such as proximal gradient descent and coordinate gradient descent are efficient but often suffer from poor local minima. In this paper, we consider a hybrid method that combines the effectiveness of combinatorial search and the efficiency of gradient descent. Specifically, we consider random strategy or greedy strategy to select a subset of coordinates as the working set, and then perform global combinatorial search over the working set based on the gradient information of the smooth objective function. The proposed method can be viewed as a block proximal Newton method. In addition, we provide some optimality analysis and convergence analysis for the proposed method. Finally, we demonstrate the efficacy of our method on some sparse optimization and binary optimization applications. As a result, our method achieves state-of-the-art performance in terms of accuracy.

1. Introduction

In this paper, we mainly focus on the following nonconvex composite minimization problem:

$$\min_{\mathbf{x}}\ F(\mathbf{x}) \triangleq f(\mathbf{x}) + h(\mathbf{x}),\ f(\mathbf{x}) \triangleq \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \langle \mathbf{x}, \mathbf{p}\rangle, \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{p} \in \mathbb{R}^n$, $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is a symmetric (not necessarily positive semidefinite) matrix, and $h(\mathbf{x})$ is a piecewise separable function [43] with $h(\mathbf{x}) = \sum_i^n h_i(\mathbf{x}_i)$. We consider two cases for $h(\cdot)$:

$$\{h_{\mathrm{b}}(\mathbf{x}) \triangleq I_\Psi(\mathbf{x})\} \ \text{ or } \ \{h_{\mathrm{s}}(\mathbf{x}) \triangleq \lambda\|\mathbf{x}\|_0 + I_\Omega(\mathbf{x})\},$$

where $\Psi \triangleq \{-1,1\}^n$, $\Omega \triangleq [-\rho\mathbf{1}, \rho\mathbf{1}]$, $I_\Psi(\cdot)$ is an indicator function on Ψ with $I_\Psi(\mathbf{x}) = \left\{ \begin{smallmatrix} 0, & \mathbf{x}\in\Psi; \\ +\infty, & \mathbf{x}\notin\Psi. \end{smallmatrix} \right.$, $\|\cdot\|_0$ is a function that counts the number of nonzero elements in a vector, λ and ρ are strictly positive constants. When $h \triangleq h_{\mathrm{b}}$, (1) refers to the binary optimization problem; when $h \triangleq h_{\mathrm{s}}$, (1) corresponds to the sparse optimization problem. We note that the feasible solution set in (1) is compact.

Binary optimization and sparse optimization capture a variety of applications of interest in both machine learning and computer vision, including binary hashing [44, 45], dense subgraph discovery [53, 51], Markov random fields [8], compressive sensing [11, 17], sparse coding [25, 1, 2], subspace clustering [18], to name only a few. In addition, binary optimization and sparse optimization are closely related to each other. A binary optimization problem can be reformulated as a sparse optimization problem using the fact that [50, 49]: $\mathbf{x} \in \{-1,1\}^n \Leftrightarrow \|\mathbf{x}-\mathbf{1}\|_0 + \|\mathbf{x}+\mathbf{1}\|_0 \leq n$, and the reverse is also true using the variational reformulation of ℓ_0 norm [6]: $\forall \|\mathbf{x}\|_\infty \leq \rho,\ \|\mathbf{x}\|_0 = \min\ \langle \mathbf{1}, \mathbf{v}\rangle,\ s.t.\ \mathbf{v} \in \{0,1\}^n,\ |\mathbf{x}| \leq \rho\mathbf{v}$. There are generally four classes of methods for solving the binary or sparse optimization problem in the literature, which we present below.

Relaxed Approximation Method. One popular method to solve (1) is convex or nonconvex relaxed approximation method. Box constrained relaxation, semi-definite programming relaxation and spherical relaxation are often used for solving binary optimization problems, while ℓ_1 norm, top-k norm, Schatten ℓ_p norm and others (such as re-weighted ℓ_1 norm, capped ℓ_1, half quadratic et al.) are often used for solving sparse optimization problems. It is generally accepted that nonconvex method often achieves better accuracy than the convex counterpart. Despite its merits, this class of method fails to directly control the sparse or binary property of the solution.

Greedy Pursuit Method. This method often solves cardinality constrained discrete optimization problems. For sparse optimization, this method greedily selects at each step one atom of the variables which have some desirable benefits [42, 15, 33, 7, 32]. It has a monotonically decreasing property and optimality guarantees in some situations, but it is limited to solving problems with smooth objective functions (typically the square function). For binary optimization, this method is strongly related to submodular optimization [10] as minimizing a set function can be reformulated as a binary optimization problem.

Combinatorial Search Method. Combinatorial search method [14] is typically concerned with problems that are NP-hard. A naive method is exhaustive search (a.k.a generate and test method). It systematically enumerates all possible candidates for the solution and pick the best candidate corresponding to the lowest objective value. The cutting plane method solves the convex linear programming relaxation and adds linear constraints to drive the solution towards binary variables. The branch-and-cut method works by performing branches and applying cuts at the nodes of the tree having a lower bound that is worse than the current solution. It maintains a probable upper bound and lower bound on the globally optimal objective value and terminates with some guarantees. Although in some cases the cutting plane method and branch-and-cut method converge without much effort, in the worse case they end up solving all 2^n convex subproblems.

Proximal Point Method. Based on the current gradient $\nabla f(\mathbf{x}^k)$, proximal point method [3, 28, 23, 36, 37, 38, 26] iteratively performs a gradient update followed by a proximal operation: $\mathbf{x}^{k+1} = \text{prox}_{\gamma h}(\mathbf{x}^k - \gamma \nabla f(\mathbf{x}^k))$. Here the proximal operator $\text{prox}_{\tilde{h}}(\mathbf{a}) = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{x}-\mathbf{a}\|_2^2 + \tilde{h}(\mathbf{x})$ can be evaluated analytically, $\gamma = 1/L$ is the step size with L being the Lipschitz constant. This method is closely related to coordinate descent [34, 12, 9] and block coordinate descent [16, 40, 5, 20, 29, 48] in the literature. Due to its simplicity, many strategies (such as variance reduction [24, 47, 13], asynchronous parallelism [27, 41], non-uniform sampling [54]) have been proposed to accelerate the proximal point algorithm. However, existing works use a scalar step size and solve a first-order majorization/surrogate function via closed form updates. Since problem (1) is nonconvex, the scaled identity quadratic majorization function may not necessarily be a good approximation for the original problem.

Recently, the authors of [52] considers a matrix splitting method for composite function minimization. Incorporating with a new triangle proximal operator procedure, the matrix splitting method achieves state-of-the-art performance. One good merit of this method is to explore the specific second-order information of the problem. Inspired by this observation, we propose a hybrid method which can better capture the second-order information of the optimization problem.

Compared to existing solutions mentioned above, our method has the following merits. (i) It can directly control the sparse or binary property of the solution. (ii) It is a greedy gradient descent algorithm [1]. (iii) It combines the effectiveness of combinatorial search. (iv) It significantly outperforms proximal point method and inherits its computational advantages.

The contributions of this paper are three-fold. (i) Algorithmically, we introduce a novel hybrid method (denoted as HYBRID) for sparse or binary optimization which combines the effectiveness of combinatorial search and the efficiency of gradient descent (See Section 2). (ii) Theoretically, we establish the optimality hierarchy and the convergence rate of our proposed algorithm (See Section 3). Our algorithm finds a stronger stationary point than existing methods. (iii) Empirically, we have conducted extensive experiments on some some binary optimization and sparse optimization tasks to show the superiority of our method (See Section 4).

2. Proposed Algorithm

This section presents our hybrid method for solving the optimization problem in (1). It can be viewed as a block proximal Newton method. Following the approach of [43], we keep the non-smooth non-convex function $h(\mathbf{z})$ and build a quadratic Newton approximation around any solution $\mathbf{x}^t$ for the smooth objective function $f(\mathbf{x})$ by considering its second-order Taylor expansion:

$$\begin{aligned} \mathcal{T}(\mathbf{z}, \mathbf{x}^t) \triangleq \; & \tfrac{1}{2}(\mathbf{z}-\mathbf{x}^t)^T \nabla^2 f(\mathbf{x}^t)(\mathbf{z}-\mathbf{x}^t) \\ & + \langle \mathbf{z}-\mathbf{x}^t, \nabla f(\mathbf{x}^t)\rangle + h(\mathbf{z}) \end{aligned} \tag{2}$$

where $\nabla f(\mathbf{x}^t)$ and $\nabla^2 f(\mathbf{x}^t)$ denote the first-order and second-order gradient of f, respectively. Our method is an iterative procedure. In every iteration, the index set of variables is separated to two sets B and N, where B is the working set. Each time variables corresponding to N are fixed while a sub-problem on variables corresponding to B is minimized. We use $\mathbf{x}_B$ to denote the sub-vector of $\mathbf{x}$ indexed by B. The proposed method is summarized in Algorithm 1.

At first glance, Algorithm 1 might seem to be merely a simple block coordinate gradient algorithm [43] applied to problem (1). However, it has some interesting properties that are worth commenting on.

• **Two New Ingredients.** The proposed algorithm has two new ingredients. (i) Instead of using majorization techniques for optimizing the block of variable, we consider the original second-order gradient information. Although the subproblem is NP-hard and admits no closed form solution, we can consider an exhaustive search to solve it exactly. (ii) We introduce a new proximal point strategy for the subproblem. This is to guarantee sufficient descent condition of the objective function and global convergence of the proposed algorithm (refer to Lemma 1 and Theorem 1).

[1] This is in contract with greedy pursuit method where the solutions must be initialized to zero and may cause divergence when being incorporated to solve bilinear matrix factorization [2].

Algorithm 1 A Hybrid Approach for Sparse or Binary Optimization

1: Input: the size of the working set k, an initial feasible solution $\mathbf{x}^0$. Set $t = 0$.
2: **while** not converge **do**
3: (S1) Employ some strategy to find a working set B of size k. Denote $N \triangleq \{1, ..., n\} \setminus B$.
4: (S2) Solve the following subproblem *globally* using combinatorial search:

$$\mathbf{x}^{t+1} \Leftarrow \arg\min_{\mathbf{z}} \ \mathcal{T}(\mathbf{z}, \mathbf{x}^t) + \frac{\theta}{2}\|\mathbf{z} - \mathbf{x}^t\|^2 \quad s.t. \ \mathbf{z}_N = \mathbf{x}^t_N \tag{3}$$

5: (S3) Increment t by 1
6: **end while**

• **Solving the Subproblem Globally.** Problem (3) in Algorithm 1 is equivalent to the following small-sized composite optimization problem: $\min_{\mathbf{z}_B} \ \frac{1}{2}(\mathbf{z} - \mathbf{x}^t)_B^T \mathbf{Q}_{B,B}(\mathbf{z} - \mathbf{x}^t)_B + \langle (\mathbf{z} - \mathbf{x}^t)_B, (\nabla f(\mathbf{x}^t))_B \rangle + h(\mathbf{z}_B) + \frac{\theta}{2}\|\mathbf{z}_B - \mathbf{x}^t_B\|_2^2$. By rearranging terms we obtain the following equivalent optimization problem:

$$\min_{\mathbf{z} \in \mathbb{R}^k} \ \frac{1}{2}\mathbf{z}^T \bar{\mathbf{Q}} \mathbf{z} + \langle \mathbf{z}, \bar{\mathbf{p}} \rangle + h(\mathbf{z}) \tag{4}$$

with suitable $\bar{\mathbf{Q}} \in \mathbb{R}^{k \times k}$ and $\bar{\mathbf{p}} \in \mathbb{R}^k$. For both binary optimization and sparse optimization, the problem in (4) can be reformulated as a integer/mixed-integer optimization problem and solved by a branch-and-bound solver such as CPLEX. For simplicity, we use a simple exhaustive search to solve it. For every variable $\mathbf{z}_i$, $i = 1, 2, ..., k$, it has two states, i.e., $-1/+1$ and zero/nonzero. We systematically enumerate the full binary tree for $\mathbf{z}$ to obtain all possible candidate solutions and then pick the best one that leads to the lowest objective value as the optimal solution.

• **Finding a Working Set.** We observe that it contains C_n^k possible combinations of choice for the working set. One may use a cyclic strategy to alternatingly select all the choices of the working set. However, past results show that coordinate gradient method results in faster convergence when the working set in selected in an arbitrary order [21] or in a greedy manner [22]. This inspires us to use random strategy and greedy strategy for finding the working set in Algorithm 1. We remark that the combination of the two strategies is preferred in practice.

Random strategy. We uniformly select one combination (which contains k coordinates) from the whole working set of size C_n^k. One remarkable benefit of this strategy is that our algorithm is ensured to find the block-k stationary point in expectation.

Greedy strategy. We use the following variable selection strategy for sparse optimization, and similar strategy can be directly applied to the binary case. Generally speaking, we pick top-k coordinates that lead to the greatest descent when one variable is changed and the rest variables are fixed based on the current solution $\mathbf{x}^t$. We denote $I \triangleq \{i : \mathbf{x}^t_i = 0\}$ and $J \triangleq \{j : \mathbf{x}^t_j \neq 0\}$. We expect the working set is balanced and pick $k/2$ coordinates from I and $k/2$ coordinates from J. For I, we solve a one-variable subproblem to compute the possible decrease for all $i \in I$ of $\mathbf{x}^t$ when changing from zero to nonzero:

$$\forall i = 1, ..., |I|, \ \mathbf{c}_i = \min_\alpha F(\mathbf{x}^t + \alpha \mathbf{e}_i) - F(\mathbf{x}^t) \tag{5}$$

For J, we compute the decrease for each coordinate $j \in J$ of $\mathbf{x}^t$ when changing from nonzero to exactly zero:

$$\forall j = 1, ..., |J|, \ \mathbf{d}_j = F(\mathbf{x}^t + \alpha \mathbf{e}_j) - F(\mathbf{x}^t), \ \alpha = \mathbf{x}^t_j \tag{6}$$

Here $\mathbf{e}_i$ is a unit vector with a 1 in the ith entry and 0 in all other entries. We sort the vectors $\mathbf{c}$ and $\mathbf{d}$ in increasing order and then pick top-$(k/2)$ coordinates form I and top-$(k/2)$ coordinates from J as the working set. If either $|I| < k/2$ or $|J| < k/2$, one can pick the whole set of I or J as a part of the working set. Here we assume that k is an even number. We remark that using the structure of $f(\cdot)$ and $h(\cdot)$, we can further simplify c_i and d_j as $\mathbf{c}_i = \alpha(\mathbf{Q}\mathbf{x} + \mathbf{p})_i + 0.5\alpha^2 \mathbf{Q}_{i,i} + \lambda$ and $\mathbf{d}_j = \alpha(\mathbf{Q}\mathbf{x} + \mathbf{p})_j + 0.5\alpha^2 \mathbf{Q}_{j,j} - \lambda$, respectively.

• **Computational Iteration Complexity.** In each iteration, solving the NP-hard sub-problem, reconstructing the gradient (using previous gradient information), and computing the partial Hessian matrix take $\mathcal{O}(2^k)$, $\mathcal{O}(nk)$, and k^2 flops, respectively. Assuming it takes $\#it$ for Algorithm 1 to converge, we have the following computational complexity for Algorithm 1: $\#it \times (\mathcal{O}(nk) + \mathcal{O}(2^k))$.

• **Extensions to Cardinality Constrained Problems.** In many applications, it is desirable to directly control the cardinality of the solution using the following constraints:

$$\{h_{\text{bc}}(\mathbf{x}) \triangleq I_\Upsilon(\mathbf{x})\} \ \text{ or } \ \{h_{\text{sc}}(\mathbf{x}) \triangleq I_\Phi(\mathbf{x})\},$$

where $\Upsilon \triangleq \{\mathbf{x} \mid \mathbf{x} \in \{0,1\}^n, \ \mathbf{x}^T\mathbf{1} = s\}$, $\Phi \triangleq \{\mathbf{x} \mid \|\mathbf{x}\|_0 \leq s\}$. The proposed block coordinate method (including the exhaustive search algorithm and the working set selection strategies) can still be applied even when $h(\mathbf{x})$ contains one non-separable constraint. What one needs is to ensure that the solution $\mathbf{x}^t$ is a feasible solution for all $t = 0, 1, ...\infty$. This is similar to the prior work of [31].

3. Theoretical Analysis

This section provides some optimality analysis and convergence analysis of our method.

3.1. Optimality Analysis

In the sequel, we present some necessary optimal conditions for (1). Since the block-k optimality is novel in this paper, it is necessary to clarify formally its relations with existing optimality conditions.

Definition 1. *(Basic Stationary Point) A solution $\breve{\mathbf{x}}$ is called a basic stationary point if the following holds. $h \triangleq h_b : \breve{\mathbf{x}} \in \{-1,+1\}^n$; $h \triangleq h_s : \breve{\mathbf{x}}_S = \arg\min_{\mathbf{z}\in[-\rho\mathbf{1},\rho\mathbf{1}]} \frac{L}{2}\|\mathbf{z} - (\breve{\mathbf{x}} - \nabla f(\breve{\mathbf{x}})/L)_S\|_2^2$. Here, L is the gradient Lipschitz constant of $f(\cdot)$ and $S \triangleq \{i|\breve{\mathbf{x}}_i \neq 0\}$.*

Remarks: For binary optimization, any binary solution is a basic stationary point. For sparse optimization, basic stationary point states that the solution achieves its global optimality when the support set is restricted. One remarkable feature of the basic stationary condition is that the solution set is enumerable and its size is 2^n. It makes it possible to validate whether a solution is optimal for the original discrete optimization problem.

Definition 2. *(L-Stationary Point) A solution $\dot{\mathbf{x}}$ is called an L-stationary point if the following condition holds: $\dot{\mathbf{x}} = \arg\min_{\mathbf{z}} \frac{L}{2}\|\mathbf{z} - (\dot{\mathbf{x}} - \nabla f(\dot{\mathbf{x}})/L)\|_2^2 + h(\mathbf{z})$. Here, L is the gradient Lipschitz constant of $f(\cdot)$.*

Remarks: This is the well-known proximal thresholding operator [3]. Although it has a closed-form solution, this scaled identity quadratic function may not be a good majorization/surrogate function for the non-convex problem.

Definition 3. *(Block-k Stationary Point) A solution $\bar{\mathbf{x}}$ is a block-k stationary point if the following condition holds: $\bar{\mathbf{x}}_B \in \arg\min_{\mathbf{z}} \mathcal{P}(\mathbf{z};\bar{\mathbf{x}},B) \triangleq \frac{1}{2}(\mathbf{z} - \bar{\mathbf{x}}_B)^T(\nabla^2 f(\bar{\mathbf{x}}))_{BB}(\mathbf{z} - \bar{\mathbf{x}}_B) + \langle \mathbf{z} - \bar{\mathbf{x}}_B, (\nabla f(\bar{\mathbf{x}}))_B\rangle + h(\mathbf{z})$, $\forall B$ with $|B| = k$.*

Remarks: Block-k stationary point is novel in this paper. One remarkable feature of this concept is that it involves solving a small-sized NP-hard problem which can be tackled by some practical global optimization method.

The following proposition states the relations between the three types of stationary point.

Proposition 1. ***Proof of the Hierarchy between the Necessary Optimality Conditions.*** *We have the following optimality hierarchy:* $\boxed{\text{Basic Stat. Point}} \overset{(1)}{\Leftarrow} \boxed{\text{L-Stat. Point}} \overset{(2)}{\Leftarrow} \boxed{\text{Block-1 Stat. Point}} \overset{(3)}{\Leftarrow} \boxed{\text{Block-2 Stat. Point}} \Leftarrow \ldots \Leftarrow \boxed{\text{Block-}n\text{ Stat. Point}} \overset{(4)}{\Leftrightarrow} \boxed{\text{Optimal Point}}$.

Proof. We use $\breve{\mathbf{x}}$, $\dot{\mathbf{x}}$, and $\bar{\mathbf{x}}$ to denote a basic stationary point, an L-stationary point, and a block-k stationary point, respectively. For block-k stationary point, we assume that the parameter k is known from the context. We denote $\Pi(\mathbf{a}) \triangleq \min(\rho\mathbf{1}, \max(-\rho\mathbf{1}, \mathbf{a}))$.

(1) We now prove that an L-stationary point $\dot{\mathbf{x}}$ is also a basic stationary point $\breve{\mathbf{x}}$. For binary optimization, this conclusion clearly holds. We now consider sparse optimization (i.e. $h \triangleq h_s$). Observing that the problem (see Definition 1) is separable, we have the following closed form solution: $\breve{\mathbf{x}}_S = \Pi(\breve{\mathbf{x}}_S - (\nabla f(\breve{\mathbf{x}}))_S/L)$. For the problem in Definition (2), we have the following closed form solution for $\dot{\mathbf{x}}$: $\dot{\mathbf{x}}_i = \begin{cases} \Pi(\dot{\mathbf{x}}_i - \nabla_i f(\dot{\mathbf{x}})/L), & (\dot{\mathbf{x}}_i - \nabla_i f(\dot{\mathbf{x}})/L)^2 > 2\lambda/L; \\ 0, & (\dot{\mathbf{x}}_i - \nabla_i f(\dot{\mathbf{x}})/L)^2 \leq 2\lambda/L. \end{cases}$ Clearly, the latter formulation implies the former one. Moreover, it is not hard to notice that for all i that $\dot{\mathbf{x}}_i = 0$, we have $|\nabla_i f(\dot{\mathbf{x}})| \leq \sqrt{2\lambda L}$, and for all j that $\dot{\mathbf{x}}_j \neq 0$, we have $\nabla_j f(\dot{\mathbf{x}}) = 0$ and $|\dot{\mathbf{x}}_j| \geq \min(\rho, \sqrt{2\lambda/L})$.

(2) Note that the minimization problem in Definition 3 is separable when $k = 1$. For $h = h_{\rm b}$, we have the following result: $\bar{\mathbf{x}}_i = \left\{ \begin{array}{ll} 1, & \bar{\mathbf{x}}_i - \nabla_i f(\bar{\mathbf{x}})/\mathbf{Q}_{i,i} > 0; \\ -1, & \bar{\mathbf{x}}_i - \nabla_i f(\bar{\mathbf{x}})/\mathbf{Q}_{i,i} \leq 0. \end{array} \right\}$. For $h = h_{\rm s}$, we have the following result: $\bar{\mathbf{x}}_i = \left\{ \begin{array}{ll} \Pi(\bar{\mathbf{x}}_i - \nabla_i f(\bar{\mathbf{x}})/\mathbf{Q}_{i,i}), & (\bar{\mathbf{x}}_i - \nabla_i f(\bar{\mathbf{x}})/\mathbf{Q}_{i,i})^2 > 2\lambda/\mathbf{Q}_{i,i}; \\ 0, & (\bar{\mathbf{x}}_i - \nabla_i f(\bar{\mathbf{x}})/\mathbf{Q}_{i,i})^2 \leq 2\lambda/\mathbf{Q}_{i,i}. \end{array} \right\}$. Since $\mathbf{Q}_{i,i} \leq L$ for all i, we conclude that block-1 stationary point implies L-stationary point.

(3) We now show that block-k_1 stationary point implies block-k_2 stationary point when $k_1 \geq k_2$. Note that to guarantee block-k stationary condition, one need to solve the problem in Definition 3 for $\sum_{i=0}^{k} C_n^k$ times, i.e. all the combination which is at most of size k. Clearly, when $k_1 \geq k_2$, the subproblems for block-k_2 stationary point is a subset of those of block-k_1 stationary point.

(4) When $B = \{1, 2, ..., n\}$, we have the following problem: $\min_{\mathbf{z}} \frac{1}{2}(\mathbf{z}-\bar{\mathbf{x}})^T\mathbf{Q}(\mathbf{z}-\bar{\mathbf{x}}) + \langle \mathbf{z}-\bar{\mathbf{x}}, \mathbf{Q}\bar{\mathbf{x}}+\mathbf{p}\rangle + \lambda\|\mathbf{z}\|_0$. After rearranging terms, this optimization problem is completely equivalent to the original problem as in (1).

□

Remarks: It is worthwhile to point out that the seminal work of [3] also presents an optimality condition for sparse optimization. However, our block-k condition is stronger than their coordinate-wise optimality since their optimal condition corresponds to $k = 1$ in our optimality condition framework.

	Basic-Stat.	L-Stat.	Block-1 Stat.	Block-2 Stat.	Block-3 Stat.	Block-4 Stat.	Block-5 Stat.	Block-6 Stat.
$h \triangleq h_{\rm b}$	64	56	9	3	1	1	1	1
$h \triangleq h_{\rm s}$	64	58	11	2	1	1	1	1

Table 1: Number of points satisfying optimality conditions.

A Running Example. We consider the quadratic optimization problem $\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{p} + h(\mathbf{x})$ where $n = 6$, $\mathbf{Q} = \mathbf{c}\mathbf{c}^T + \mathbf{I}$, $\mathbf{p} = \mathbf{1}$, $\mathbf{c} = [1\ 2\ 3\ 4\ 5\ 6]^T$ when $h(\cdot) \triangleq h_{\rm s}(\mathbf{x})$ and $h(\cdot) \triangleq h_{\rm b}(\mathbf{x})$. The parameters for $h_{\rm s}(\mathbf{x})$ are set to $\lambda = 0.01$ and $\rho = +\infty$. The stationary point distribution on this example can be found in Table 1. This problem contains $\sum_{i=0}^{6} C_6^i = 64$ stationary points. There are 56 and 58 local minimizers satisfying the L-stationary condition for binary optimization and sparse optimization problem, respectively. There are 9 and 11 local minimizers satisfying the block-1 stationary condition. Moreover, as k becomes large, the newly introduced type of local minimizers (i.e. block-k stationary point) become more restricted in the sense that they have small number of stationary points.

3.2. Convergence Analysis

This subsection provides some convergence analysis for Algorithm 1. We assume that the random strategy is used for finding the working set. We show that our method converges to the block-k stationary point for discrete optimization in expectation. More convergence analysis can be found in the **Appendix**.

The following sufficient decrease condition is useful in our proof.

Lemma 1. *(**Sufficient Decrease Condition**) Assume B is the working set in the t-th iteration. Suppose $\{F(x^t)\}_{t=0}^{\infty}$ is generated by Algorithm 1, the following inequality holds:*

$$F(\mathbf{x}^{t+1}) - F(\mathbf{x}^t) \leq -\frac{\theta}{2}\|\mathbf{x}_B^{t+1} - \mathbf{x}_B^t\|^2 \quad (7)$$

Proof. We define $N \triangleq \{1, 2, ..., n\} \setminus B$ and $\mathbf{s} \triangleq \mathbf{x}^{t+1} - \mathbf{x}^t$. In the t-th iteration, since we solve the optimization problem in (3), we have $T(\mathbf{x}^{t+1}, \mathbf{x}^t) + \frac{\theta}{2}\|\mathbf{x}^{t+1} - \mathbf{x}^t\|^2 \leq T(\mathbf{x}, \mathbf{x}^t) + \frac{\theta}{2}\|\mathbf{x} - \mathbf{x}^t\|^2$, $\forall \mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x}_N = \mathbf{x}_N^t$. Letting $\mathbf{x} = \mathbf{x}^t$, we have: $\frac{1}{2}(\mathbf{x}^{t+1} - \mathbf{x}^t)^T\mathbf{Q}(\mathbf{x}^{t+1} - \mathbf{x}^t) + (\mathbf{x}^{t+1} - \mathbf{x}^t)^T\nabla f(\mathbf{x}^t) + h(\mathbf{x}^{t+1}) + \frac{\theta}{2}\|\mathbf{x}^{t+1} - \mathbf{x}^t\|^2 \leq h(\mathbf{x}^t)$.

One the other hand, using the specific structure of $f(\cdot)$, we obtain the following results: $\frac{1}{2}(\mathbf{x}^{t+1} - \mathbf{x}^t)^T\mathbf{Q}(\mathbf{x}^{t+1} - \mathbf{x}^t) + (\mathbf{x}^{t+1} - \mathbf{x}^t)^T\nabla f(\mathbf{x}^t) = f(\mathbf{x}^{t+1}) - f(\mathbf{x}^t)$. Thus, we we naturally have the following result: $F(\mathbf{x}^{t+1}) - F(\mathbf{x}^t) \leq -\frac{\theta}{2}\|\mathbf{x}_B^{t+1} - \mathbf{x}_B^t\|_2^2$, where we use the fact that $\mathbf{s}_N = \mathbf{0}$. Thus, we finish the proof of this lemma.

□

Remarks: The introduction of the strongly convex parameter $\theta > 0$ is necessary for our nonconvex optimization problem since it guarantees sufficient decrease condition which is important for convergence.

Theorem 1. ***Convergence Properties for** $h \triangleq h_s$ **or** $h \triangleq h_b$.* *Let $\mathbf{x}^t$ be the sequence generated by Algorithm 1. Assume that the working set of size k is selected uniformly. We have the following results. (i) It holds that $\lim_{t\to\infty} \mathbb{E}[\|\mathbf{x}^{t+1} - \mathbf{x}^t\|] = 0$. (ii) As $t \to \infty$, $\mathbf{x}^t$ converges to the block-k stationary point $\bar{\mathbf{x}}$ of (1) in expectation. (iii) For $h \triangleq h_b$, we have $\mathbb{E}[\|\mathbf{x}^{t+1} - \mathbf{x}^t\| \mid \mathbf{x}^t] \geq \sqrt{2}/C_n^k$. The solution changes at most $\sqrt{2}C_n^k(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))/\theta$ times in expectation for finding a block-k stationary point. (iv) For $h \triangleq h_s$, we have $|\mathbf{x}_i^t| \geq \delta$ for all i with $\mathbf{x}_i^t \neq 0$, where $\delta \triangleq \min_j \ \{\min(\rho, |\mathbf{x}_j^0|, \sqrt{2\lambda/(\theta + \mathbf{Q}_{j,j})})\}$. Moreover, it holds that: $\mathbb{E}[\|\mathbf{x}^{t+1} - \mathbf{x}^t\| \mid \mathbf{x}^t] \geq \delta/C_n^k$. The solution changes at most $2C_n^k(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))/(\theta\delta)$ times in expectation for finding a block-k stationary point.*

Proof. (i) Firstly, taking the expectation of B for the sufficient descent inequality in Lemma 1, we have

$$\mathbb{E}[F(\mathbf{x}^{t+1}) \mid \mathbf{x}^t] \leq F(\mathbf{x}^t) - \mathbb{E}[\tfrac{\theta}{2}\|\mathbf{x}^{t+1} - \mathbf{x}^t\|\mathbf{x}^t] \quad (8)$$

Summing (8) over $i = 0, 1, 2, ..., t-1$, we have:

$$\begin{aligned} &\tfrac{\theta}{2}\textstyle\sum_{i=0}^t \mathbb{E}[\|\mathbf{x}^{i+1} - \mathbf{x}^i\|_2^2 \mid \mathbf{x}^i] \leq F(\mathbf{x}^0) - F(\mathbf{x}^t) \\ \Rightarrow &\tfrac{\theta}{2}\textstyle\sum_{i=0}^t \mathbb{E}[\|\mathbf{x}^{i+1} - \mathbf{x}^i\|_2^2 \mid \mathbf{x}^i] \leq F(\mathbf{x}^0) - F(\bar{\mathbf{x}}) \\ \Rightarrow &\min_{i=1,...,t} \mathbb{E}[\tfrac{1}{2}\|\mathbf{x}^{i+1} - \mathbf{x}^i\|_2^2 \mid \mathbf{x}^i] \leq \frac{F(\mathbf{x}^0) - F(\bar{\mathbf{x}})}{t\theta} \end{aligned} \quad (9)$$

where the second conclusion uses the fact that $F(\bar{\mathbf{x}}) \leq F(\mathbf{x}^t)$. Therefore, we have $\lim_{t\to\infty} \mathbb{E}[\|\mathbf{x}^{t+1} - \mathbf{x}^t\| \mid \mathbf{x}^t] = 0$.

(ii) We assume the conclusion does not hold. In expectation there exists a block of coordinates B such that $\mathbf{x}_B^t \notin \arg\min_{\mathbf{z}} \ \mathcal{P}(\mathbf{z}; \mathbf{x}^t, B)$, where $\mathcal{P}(\cdot)$ is defined in Definition 3. However, according to the fact that $\mathbf{x}^t = \mathbf{x}^{t+1}$ and step S2 in Algorithm 1, we have $\mathbf{x}_B^{t+1} \in \arg\min_{\mathbf{z}} \ \mathcal{P}(\mathbf{z}; \mathbf{x}^t, B)$. Hence, we have $\mathbf{x}_B^t \neq \mathbf{x}_B^{t+1}$. This contradicts with the fact that $\mathbf{x}^t = \mathbf{x}^{t+1}$ as $t \to \infty$. We conclude that $\mathbf{x}^t$ converges to the block-k stationary point.

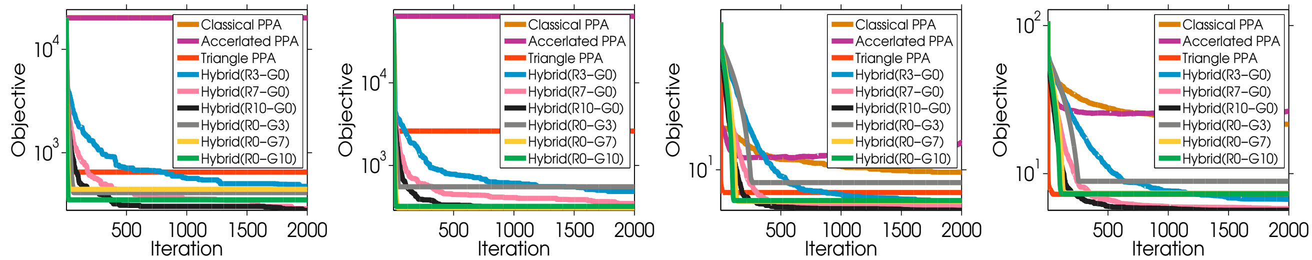
(iii) We observe that when the current solution $\mathbf{x}^t$ changes, we have $\|\mathbf{x}^{t+1} - \mathbf{x}^t\| \geq \sqrt{2}$ if $\mathbf{x}^t \neq \mathbf{x}^{t+1}$. Noticing that there are C_n^k possible combinations of choice for the working set of size k. Thus, we have $\mathbb{E}[\|\mathbf{x}^{t+1} - \mathbf{x}^t\| \mid \mathbf{x}^t] \geq \sqrt{2}/C_n^k$. From (9), we obtain: $(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))/(t\theta) \geq \frac{\sqrt{2}}{2C_n^k}$. Therefore, the number of iterations is upper bounded by $t \leq \sqrt{2}C_n^k(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))/\theta$ times in expectation.

(iv) Note that Algorithm 1 solves the optimization problem as in (3) in every iteration. Using Proposition 1, we have that the solution $\mathbf{x}_B^{t+1}$ is also a block-1 stationary point. Therefore, we have $|\mathbf{x}^{t+1}|_i \geq \min(\rho, \sqrt{2\lambda/(\theta + \mathbf{Q}_{i,i})})$ for all $\mathbf{x}_i^{t+1} \neq 0$. Taking the initial point of $\mathbf{x}$ for consideration, we have that: $|\mathbf{x}_i^t| \geq \min(\rho, |\mathbf{x}_i^0|, \sqrt{2\lambda/(\theta + \mathbf{Q}_{i,i})})$. Therefore, we have the following results: $\|\mathbf{x}^{t+1} - \mathbf{x}^t\|_2 \geq \delta$. Taking the expectation of B, we have the following results: $\mathbb{E}[\|\mathbf{x}^{t+1} - \mathbf{x}^t\|_2 \mid \mathbf{x}^t] = \delta/C_n^k$. Moreover, we obtain: $(2F(\mathbf{x}^0) - 2F(\bar{\mathbf{x}}))/(t\theta) \geq \delta/C_n^k$ from (9). Therefore, the number of iterations is upper bounded by $t \leq 2C_n^k(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))/(\theta\delta)$. □

Remarks: (i) Our algorithm is still guaranteed to find the block-k stationary point but it is in expectation. Our proof essentially establishes the number of iterations in expectation to find the block-k stationary point. (ii) According to Theorem 1, the time complexity of the random strategy-based algorithm for finding a block-k stationary point is $C_n^k \times (\mathcal{O}(nk) + \mathcal{O}(2^k))$ in expecation.

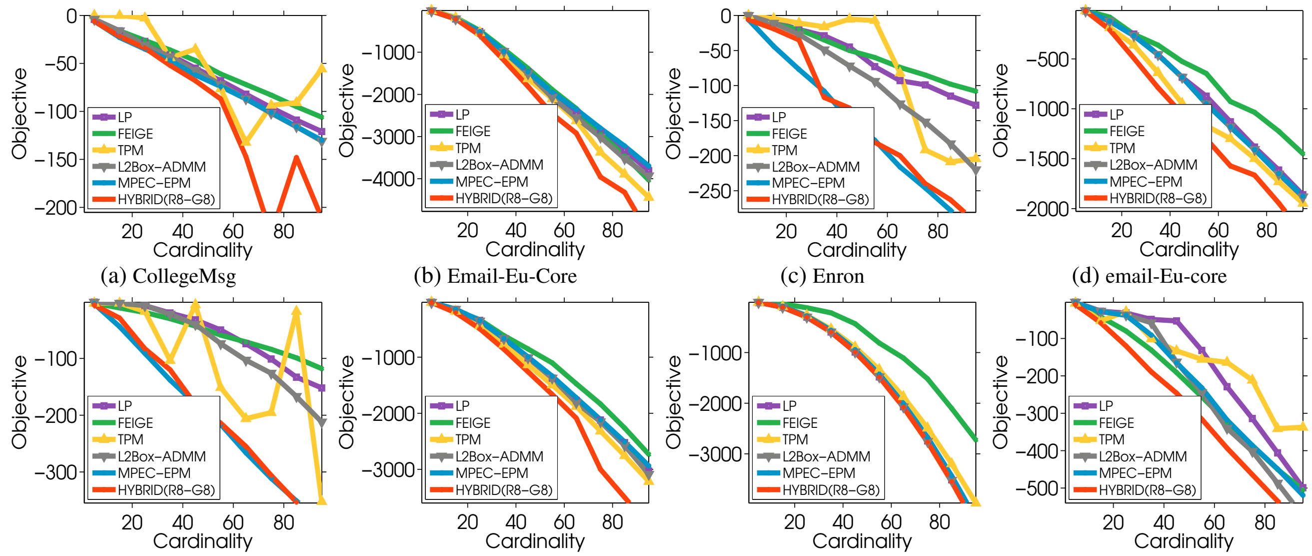
4. Experimental Validation

In this section, we demonstrate the effectiveness of our proposed algorithm on four discrete optimization tasks, namely sparse regularized least square problem, binary constrained least square problem, dense subgraph discovery, and



(a) $\sigma = 0.001$, $h = h_{\rm b}$. (b) $\sigma = 0.01$, $h = h_{\rm b}$. (c) $\sigma = 0.001$, $h = h_{\rm s}$. (d) $\sigma = 0.01$, $h = h_{\rm s}$.

Figure 1 Convergence behavior for solving the sparse regularized least squares / binary constrained least squares optimization problem with different initializations. Denoting $\tilde{\mathbf{o}}$ as an arbitrary standard Gaussian random matrix of suitable size, we consider the following strategy for different initiations $\mathbf{x}$. Figure a and b: $\mathbf{x} = sign(\sigma \times \tilde{\mathbf{o}})$. Figure c and d: $\mathbf{x} = \sigma \times \tilde{\mathbf{o}}$.



(a) CollegeMsg (b) Email-Eu-Core (c) Enron (d) email-Eu-core

(e) p2p-Gnutella04 (f) p2p-Gnutella08 (g) p2p-Gnutella09 (h) UK-2007-05

Figure 2: Experimental results on dense subgraph discovery.

sparse constrained least square problem. We use **HYBRID (Ri-Gj)** to denote our hybrid method along with selecting i coordinate using random strategy and j coordinates using the greedy strategy. Since these two strategies may select the same coordinates, the working set at most contains $i+j$ coordinates. We set $\theta = 10^{-5}$ for HYBRID. We keep a record of the relative changes of the objective function values by $r_t = (f(\mathbf{x}^t) - f(\mathbf{x}^{t+1}))/f(\mathbf{x}^t)$. We let our algorithms run up to T iterations and stop them at iteration $t < T$ if $\text{mean}([r_{t-\min(t,M)+1},\ r_{t-min(t,M)+2}, ..., r_t]) \leq \epsilon$. The defaults values of ϵ, M, and T are 10^{-5}, 50 and 1000, respectively. All codes were implemented in Matlab on an Intel 3.20GHz CPU with 8 GB RAM.

4.1. Binary Constrained / Sparse Regularized Least Squares Problem

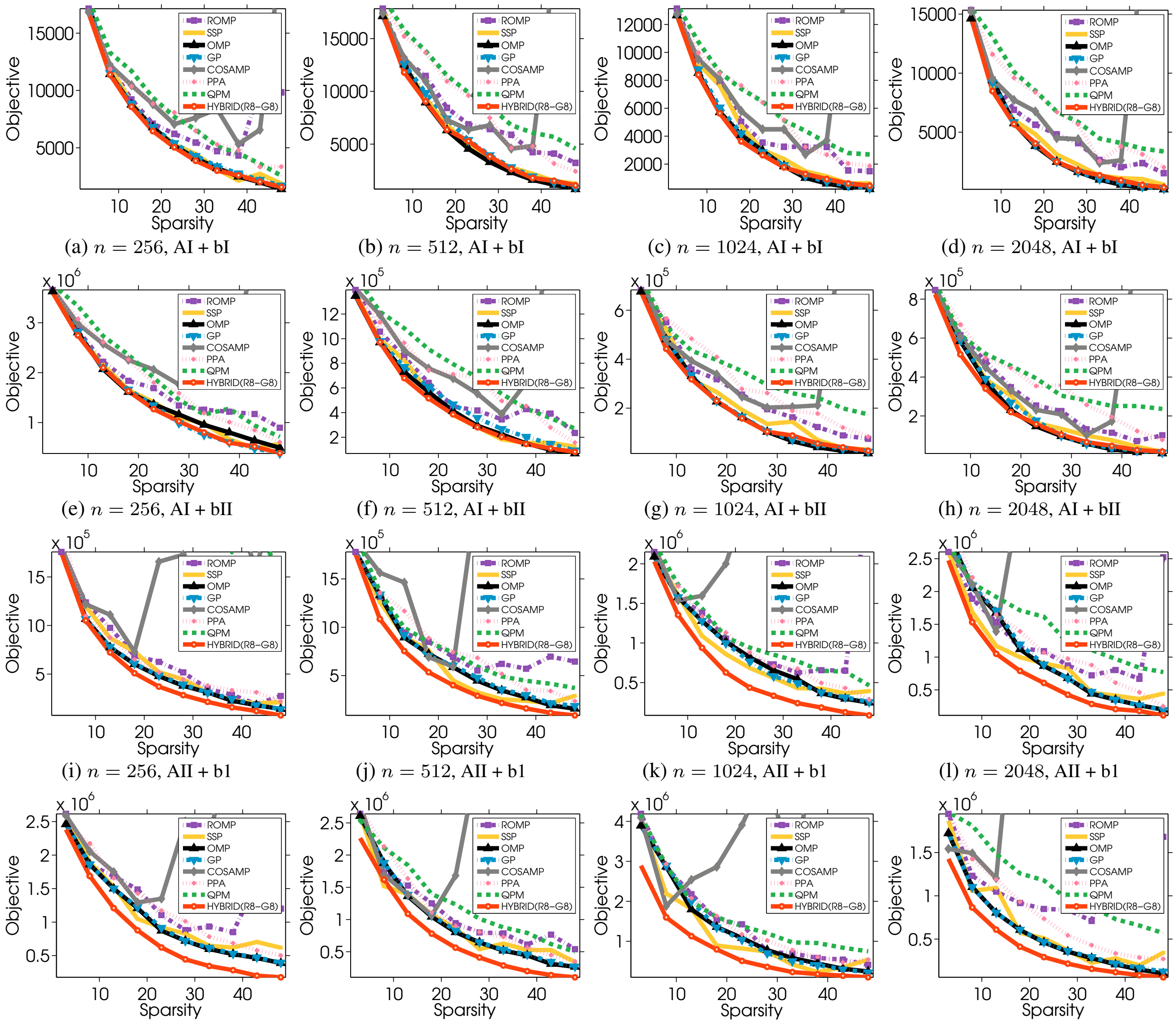
Given a design matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and an observation vector $\mathbf{b} \in \mathbb{R}^m$, sparse regularized / binary constrained least square problem is to solve the following optimization problem:

$$\min_{\mathbf{x} \in \{-1,1\}^n} \tfrac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 \text{ or } \min_{\mathbf{x}} \tfrac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_0$$

We generate $\mathbf{A} \in \mathbb{R}^{200 \times 500}$ and $\mathbf{b} \in \mathbb{R}^{200}$ from a (0-1) uniform distribution. We set $\lambda = 0.1$. Note that although HYBRID uses some randomized strategies to find the working set, we can always measure the quality of the solution by computing the deterministic objective value.

Compared Methods. We compare the proposed method (HYBRID) with three state-of-the-art methods: (i) Classical Proximal Point Algorithm (PPA) [35], (ii) accelerated PPA [35, 4], (iii) triangle PPA (also known as matrix splitting method) [52].

Experimental Results. Several observations can be drawn from Figure 1. (i) Classical PPA and accelerated PPA achieve similar performance and they get stuck into poor local minima. (ii) Triangle PPA significantly improves



(a) $n = 256$, AI + bI (b) $n = 512$, AI + bI (c) $n = 1024$, AI + bI (d) $n = 2048$, AI + bI

(e) $n = 256$, AI + bII (f) $n = 512$, AI + bII (g) $n = 1024$, AI + bII (h) $n = 2048$, AI + bII

(i) $n = 256$, AII + b1 (j) $n = 512$, AII + b1 (k) $n = 1024$, AII + b1 (l) $n = 2048$, AII + b1

(m) $n = 256$, AII + bII (n) $n = 512$, AII + bII (o) $n = 1024$, AII + bII (p) $n = 2048$, AII + bII

Figure 3: Experimental results on sparse constrained least squares problem with varying $n = \{256,\ 512,\ 1024,\ 2048\}$ and fixing $m = 512$ on different types of $\mathbf{A}$ and $\mathbf{b}$..

over PGA and APGA. This is consistent with the work of [52]. (iii) Our proposed hybrid method is more effective than triangle PPA. In addition, we find that as the parameter k becomes larger, more higher accuracy is achieved. (iii) HYBRID appears to be less sensitive to initialization and it converges to similar objective values when using different initializations. (iv) We notice that Hybrid(R0-G10) converges quickly but it generally leads to worse solution quality than Hybrid(R10-G0). Based on this observation, we consider a combined random and greedy strategies for finding the working set in our forthcoming experiments.

4.2. Dense Subgraph Discovery

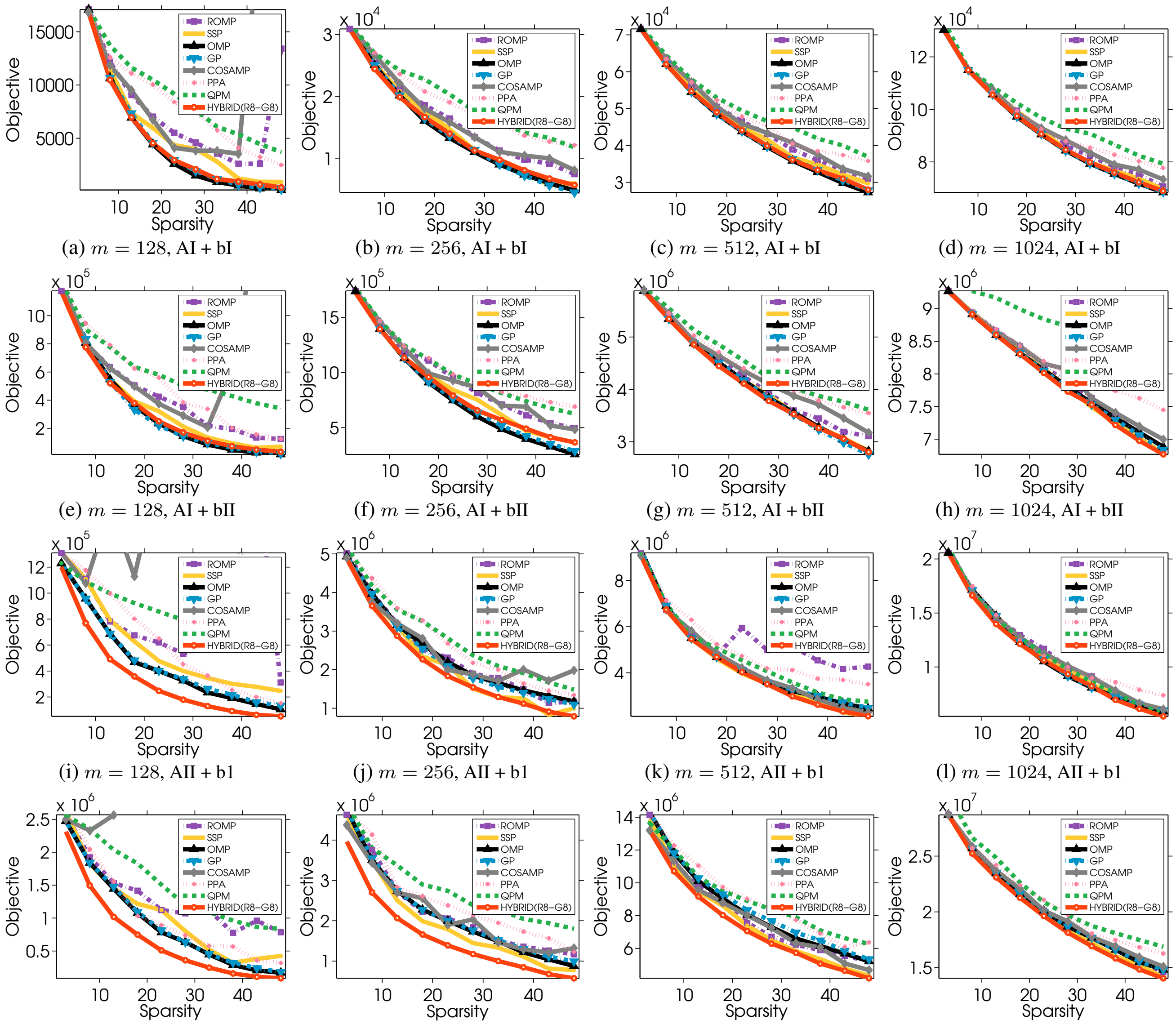
Dense subgraph discovery is an important application of binary optimization. It aims at finding the maximum density subgraph on s vertices [39, 19, 53], which can be formulated as the following binary program:

$$\min_{\mathbf{x}\in\{0,1\}^n} \ -\mathbf{x}^T\mathbf{W}\mathbf{x},\ s.t.\ \mathbf{x}^T\mathbf{1} = s, \tag{10}$$

where $\mathbf{W} \in \mathbb{R}^{n\times n}$ is the adjacency matrix of a graph.

Compared Methods. We compare our HYBRID method on eight datasets (refer to the sub-captions in Figure 2)[2] against six methods: (i) LP relaxation [51]. (ii) Feige's greedy

[2] https://snap.stanford.edu/data/



(a) $m = 128$, AI + bI (b) $m = 256$, AI + bI (c) $m = 512$, AI + bI (d) $m = 1024$, AI + bI

(e) $m = 128$, AI + bII (f) $m = 256$, AI + bII (g) $m = 512$, AI + bII (h) $m = 1024$, AI + bII

(i) $m = 128$, AII + b1 (j) $m = 256$, AII + b1 (k) $m = 512$, AII + b1 (l) $m = 1024$, AII + b1

(m) $m = 128$, AII + bII (n) $m = 256$, AII + bII (o) $m = 512$, AII + bII (p) $m = 1024$, AII + bII

Figure 4: Experimental results on sparse constrained least squares problem with varying $m = \{128,\ 256,\ 512,\ 1024\}$ and fixing $n = 2048$ on different types of $\mathbf{A}$ and $\mathbf{b}$.

algorithm (GEIGE) [19]. (iii) Truncated Power Method (TPM) [3] [53]. (iv) L2box-ADMM [46]. (v) MPEC-EPM [51]. For description of these methods, we refer to [51, 49]. We show the average results of using 3 random initial points.

Experimental Results. Several observations can be drawn from Figure 2. (i) FEIGE generally fails to solve the dense subgraph discovery problem and it leads to solutions with low density. (ii) TPM gives better performance than state-of-the-art technique MPEC-EPM in some cases but it is unstable. (iii) Our proposed HYBRID generally outperforms all the compared methods.

[3] https://sites.google.com/site/xtyuan1980/publications

4.3. Sparse Constrained Least Squares Problem

We consider the following sparse constrained least squares problem:

$$\min_{\mathbf{x}}\ \tfrac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2,\ s.t.\ \|\mathbf{x}\|_0 \leq s, \tag{11}$$

In our experiments, to generate the sparse original signal $\ddot{\mathbf{x}} \in \mathbb{R}^n$, we select a support set of size 100 uniformly at random and set them to arbitrary number sampled from standard Gaussian distribution. In order to verify the robustness of the comparing methods, we generate the design matrix $\mathbf{A}$ and the noise vector $\mathbf{o} \in \mathbb{R}^m$ with and without outliers, as follows:

$$\text{AI: } \mathbf{A} = \text{randn}(m,n), \quad \text{AII: } \mathbf{A} = \mathcal{P}(\text{randn}(m,n))$$
$$\text{bI: } \mathbf{o} = 10 \times \text{randn}(m,1), \quad \text{bII: } \mathbf{o} = \mathcal{P}(10 \times \text{randn}(m,1))$$

where randn(m,n) is a function that returns a standard Gaussian random matrix of size $m \times n$, $\mathcal{P}(\mathbf{X}) \in \mathbb{R}^{m \times p}$ denotes a noisy version of $\mathbf{X} \in \mathbb{R}^{m \times p}$ where 2% of the entries of $\mathbf{X}$ are corrupted uniformly by scaling the original values by 100 times [4]. The observation vector is generated via $\mathbf{b} = \mathbf{A}\ddot{\mathbf{x}} + \mathbf{o}$. Note that the Hessian matrix can be ill-conditioned for the 'AII' type design matrix. We vary m from $\{128,\ 256,\ \mathbf{512},\ 1024\}$ and vary n from $\{256,\ 512,\ 1024,\ \mathbf{2048}\}$. Unless otherwise specified, the default parameters in bold are used. We swap the parameter s over $\{3, 8, 13, 18...50\}$.

Compared Methods. We compare the proposed hybrid algorithm with seven state-of-the-art sparse optimization algorithms: (i) Regularized Orthogonal Matching Pursuit (ROMP) [33], (ii) Subspace Pursuit (SSP) [15], (iii) Orthogonal Matching Pursuit (OMP) [42], (iv) Gradient Pursuit (GP) [7], (v) Compressive Sampling Matched Pursuit (CoSaMP)[32], (vi) Proximal Point Algorithm (PPA) [2], and (vii) Quadratic Penalty Method (QPM) [30]. We remark that ROMP, SSP, OMP, GP and CoSaMP are greedy algorithms and their support sets need to be selected iteratively. They are non-gradient type algorithms, it is hard to incorporate these methods into other gradient-type based optimization algorithms [2]. We use the Matlab implementation in the 'sparsify' toolbox[5]. Both PPA and QPM are based on iterative hard thresholding. Since the optimal solution is expected to be sparse, we initialize the solutions of PPA, QPM and HYBRID to $10^{-12} \times \text{randn}(n,1)$ and project them to feasible solutions. The initial solution of greedy pursuit methods are initialized to zero points implicitly. We show the average results of using 3 random initial points.

Experimental Results. Several conclusions can be drawn from Figure 3 and Figure 4. (i) PPA and QPM generally lead to the worst performance. (ii) ROMP and COSAMP are not stable and sometimes they present bad performance. (iii) SSP, OMP and GP generally present comparable performance to HYBRID when the Hessian matrix is well-conditioned (for 'AI' type design matrix) but present much worse performance than HYBRID when the Hessian matrix is ill-conditioned (for 'AII' type design matrix).

4.4. Computational Efficiency of Algorithm 1

We show the convergence curve of different methods for binary optimization and sparse optimization in Figure 5. The sub-figure in the left shows our result for dense subgraph discovery problem on 'p2p-Gnutella30' dataset which contains 36682 nodes. We set k=100 in our experiments. The sub-figure in the right shows our result for sparse constrained least squares problem on 'AII'+'bII' data with $m = 512,\ n = 1024$ and $k = 20$.

Generally speaking, our HYBRID is effective and practical for large-scale discrete optimization. Although it takes longer time to converge than PPA and QPM, the computational time is acceptable and it generally takes less than 30 seconds to converge in *all* our instances. We think this computation time pays off as HYBRID achieves significantly higher accuracy than PPA and QPM. The main bottleneck of computation is on solving the small-sized subproblem using sub-exponential time $\mathcal{O}(2^k)$. The parameter k in Algorithm 1 can be viewed as a tuning parameter to balance the efficacy and efficiency. One can further accelerate the algorithm using asynchronous parallelism or mini-batch optimization techniques.

[4] Matlab script: I = randperm(m*p,round(0.02*m*p)); X(I) = X(I)*100.
[5] http://www.personal.soton.ac.uk/tb1m08/sparsify/sparsify.html

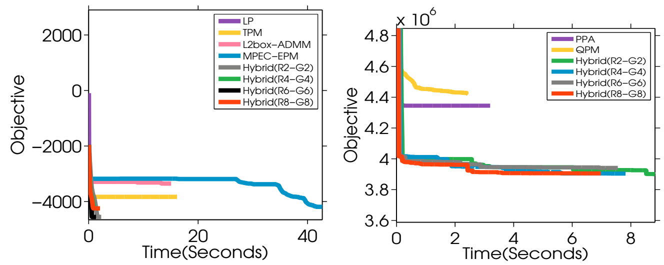


Figure 5: Convergence curve of different methods for binary optimization and sparse optimization.

5. Conclusions

This paper presents an effective and practical method for solving discrete optimization problems. Our method takes advantage of the effectiveness of combinatorial search and the efficiency of gradient descent. We also provided rigorous optimality analysis and convergence analysis for the proposed algorithm. The extensive experiments show that our method achieves state-of-the-art performance.

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Appendix

A. Proof of Convergence Rate

This section provides the proof of convergence rate of Algorithm 1 when the random strategy for finding the working set is used. We notice that the working set contains C_n^k possible different combinations and each combination contains k coordinates. We uniformly selection one combination from the working set as B_i in i-th iteration. Note that every B_i corresponds to a unique $\mathbf{U}_i$ and $\mathbf{x}_{B_i}$ can be rewritten as $\mathbf{x}_{B_i} = (\mathbf{U}_i)^T\mathbf{x}$ for some suitable binary matrix with $(\mathbf{1U})^T = \mathbf{1}$ and $\mathbf{U1} \in \{0,1\}^n$. Sometimes, we use $\mathbf{x}_i$ to denote $\mathbf{x}_{B_i}$ for brevity. It is not hard to verify that the following always holds: $\frac{1}{C_n^k}\sum_{i=1}^{C_n^k}\mathbf{U}_i\mathbf{x}_{B_i} = \frac{k}{n}\cdot\mathbf{x}$. To establish tight bound, we assume that there always exists a constant L such that $\max_{i=1}^{C_n^k}\|\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i\| \leq L$.

A.1. Convergence Rate for Binary Optimization

In what follows, we study the convergence rate of Algorithm 1 for binary optimization (i.e. $h \triangleq h_{\text{b}}$). We define $\Pi(\mathbf{a}) = \arg\min_{\mathbf{x}}\|\mathbf{x}-\mathbf{a}\|,\ s.t.\ \mathbf{x}\in\{-1,+1\}^n$. Our key observation is that when $\Pi(\mathbf{x}) \neq \Pi(\mathbf{y})$, it holds $\|\Pi(\mathbf{x})-\mathbf{x}\|_2 \leq (1-\kappa)\|\Pi(y)-\mathbf{x}\|_2$ with $0<\kappa<1$. Our analysis combines the above observation with the strongly convex property of $f(\cdot)$ to provide Q-linear convergence rate for Algorithm 1.

The following lemmas are useful in our proof.

Lemma 2. *We define* $\Pi(\mathbf{a}) = \arg\min_{\mathbf{x}}\|\mathbf{x}-\mathbf{a}\|,\ s.t.\ \mathbf{x}\in\{-1,+1\}^n$. *The following inequality always holds for all* $\mathbf{x}$ *and* $\mathbf{y}$*:*

$$\|\Pi(\mathbf{x})-\mathbf{x}\|_2^2 \leq (1-\kappa)\|\Pi(y)-\mathbf{x}\|_2^2 \tag{12}$$

with $\kappa = 0$*. Moreover, if* $\Pi(\mathbf{x}) \neq \Pi(\mathbf{y})$*, there exist a small* $0<\kappa<1$ *such that (12) holds.*

Proof. Since $\|\Pi(\mathbf{x})\|_2^2 = n$, we have the following results:

$$\begin{aligned}
& \|\Pi(\mathbf{x})-\mathbf{x}\|_2^2 \leq (1-\kappa)\|\Pi(y)-\mathbf{x}\|_2^2 \\
\Leftrightarrow\quad & \kappa\|\Pi(y)-\mathbf{x}\|_2^2 + \|\Pi(\mathbf{x})\|_2^2 + \|\mathbf{x}\|_2^2 - 2\langle\Pi(\mathbf{x}),\mathbf{x}\rangle \leq \|\Pi(y)\|_2^2 + \|\mathbf{x}\|_2^2 - 2\langle\Pi(y),\mathbf{x}\rangle \\
\Leftrightarrow\quad & \kappa\|\Pi(y)-\mathbf{x}\|_2^2 \leq 2\langle\Pi(x)-\Pi(y),\mathbf{x}\rangle
\end{aligned}$$

Note that $\Pi(\mathbf{x}) \neq \Pi(\mathbf{y})$ also implies $\mathbf{x} \neq \Pi(\mathbf{y})$ and we have $\|\Pi(y)-\mathbf{x}\|_2^2 > 0$. Moreover, we notice that $\mathbf{x}_i \cdot sign(\mathbf{y}_i) \leq \mathbf{x}_i \cdot sign(\mathbf{x}_i)$ for all i and we have $\langle\Pi(x)-\Pi(y),\mathbf{x}\rangle > 0$. Therefore, there exists a sufficient small κ such that $\kappa\|\Pi(y)-\mathbf{x}\|_2^2 \leq 2\langle\Pi(x)-\Pi(y),\mathbf{x}\rangle$ holds. This finishes the proof of this lemma. □

Lemma 3. *Assume that* $f(\cdot)$ *is* α*-strongly convex. The following inequality holds for any* $\mathbf{x}$ *and* $\mathbf{y}$*:*

$$\tfrac{\alpha^2}{4}\|\mathbf{y}-\mathbf{x}\|_2^2 - \|\nabla f(\mathbf{x})\|_2^2 \leq -\tfrac{\alpha}{2}(f(\mathbf{x})-f(\mathbf{y})) \tag{13}$$

Proof. We naturally derive the following results:

$$\begin{aligned}
& \|\nabla f(\mathbf{x})\|_2^2 - \tfrac{\alpha^2}{4}\|\mathbf{y}-\mathbf{x}\|_2^2 \\
= \quad & \left(\|\nabla f(\mathbf{x})\|_2 - \tfrac{\alpha}{2}\|\bar{\mathbf{x}}-\mathbf{x}\|_2\right)\cdot\left(\|\nabla f(\mathbf{x})\|_2 + \tfrac{\alpha}{2}\|\mathbf{y}-\mathbf{x}\|_2\right) \\
= \quad & \left(\|\nabla f(\mathbf{x})\|_2\cdot\|\mathbf{y}-\mathbf{x}\|_2 - \tfrac{\alpha}{2}\|\mathbf{y}-\mathbf{x}\|_2^2\right)\cdot\left(\|\nabla f(\mathbf{x})\|_2 + \tfrac{\alpha}{2}\|\mathbf{y}-\mathbf{x}\|_2\right)/\|\mathbf{y}-\mathbf{x}\|_2 \\
\geq \quad & \left(\langle\nabla f(\mathbf{x}),\mathbf{y}-\mathbf{x}\rangle - \tfrac{\alpha}{2}\|\mathbf{y}-\mathbf{x}\|_2^2\right)\cdot\left(\|\nabla f(\mathbf{x})\|_2 + \tfrac{\alpha}{2}\|\mathbf{y}-\mathbf{x}\|_2\right)/\|\mathbf{y}-\mathbf{x}\|_2 \\
\geq \quad & (f(\mathbf{x})-f(\mathbf{y}))\cdot(0+\tfrac{\alpha}{2}) \\
= \quad & \tfrac{\alpha}{2}(f(\mathbf{x})-f(\mathbf{y}))
\end{aligned}$$

where the first inequality uses the Cauchy-Schwarz inequality, the second inequality uses the α-strongly convexity condition that $f(\mathbf{x})-f(\mathbf{y}) \leq \langle\nabla f(\mathbf{x}),\mathbf{x}-\mathbf{y}\rangle - \frac{\alpha}{2}\|\mathbf{x}-\mathbf{y}\|_2^2$.

□

Theorem 2. ***Proof of Convergence Rate when*** $f(\cdot)$ ***is*** s***-Strongly Convex and*** $h \triangleq f_{\mathbf{b}}$***.*** *Let* $\mathbf{x}^t$ *be the sequence generated by Algorithm 1. We have the following result:*

$$E[f(\mathbf{x}^{t+1})|\mathbf{x}^t] - f(\bar{\mathbf{x}}) \leq (1-C)(f(\mathbf{x}^t) - f(\bar{\mathbf{x}})) \tag{14}$$

where $C \triangleq (\frac{1}{2(L+\theta)}\sqrt{(L+\theta-s)(1-\kappa)\kappa} + 1 - \kappa)k/n$*. In other words, it takes at most* $\log_{(1-C)}(\frac{\epsilon}{F(\mathbf{x}^0)-F(\bar{\mathbf{x}})})$ *times to find a local optimal solution satisfying* $F(\mathbf{x}^k) - F(\bar{\mathbf{x}}) \leq \epsilon$*. Here* $L \triangleq \max_{i=1}^{C_n^k} \|\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i\|$*.*

Proof. We define

$$d_i'(\mathbf{x}) = \arg\min_{\mathbf{d}\in\mathbb{R}^k} H_i'(\mathbf{x},\mathbf{d}),\ H_i'(\mathbf{x},\mathbf{d}) \triangleq f(\mathbf{x}) + \langle \nabla_i f(\mathbf{x}), \mathbf{d}_i\rangle + \tfrac{L+\theta}{2}\|\mathbf{d}_i\|_2^2 + h_{\mathbf{b}}(\mathbf{x}_i + \mathbf{d}_i)$$
$$d_i(\mathbf{x}) = \arg\min_{\mathbf{d}_i\in\mathbb{R}^k} H_i(\mathbf{x},\mathbf{d}_i),\ H_i(\mathbf{x},\mathbf{d}_i) \triangleq f(\mathbf{x}) + \langle \nabla_i f(\mathbf{x}), \mathbf{d}_i\rangle + \tfrac{1}{2}\mathbf{d}_i^T(\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i + \theta\mathbf{I})\mathbf{d}_i + h_{\mathbf{b}}(\mathbf{x}_i + \mathbf{d}_i)$$

Since each block i is picked randomly with probability $1/C_n^k$, we have the following inequalities:

$$\begin{aligned}
& \mathbb{E}[F(\mathbf{x}^{t+1}) \mid \mathbf{x}^t] = \mathbb{E}[F(\mathbf{x}^t + \mathbf{U}_i\mathbf{d}_i(\mathbf{x}^t))] \\
\leq\ & \mathbb{E}[f(\mathbf{x}^t) + \langle \nabla_i f(\mathbf{x}^t), d_i(\mathbf{x}^t)\rangle + \tfrac{1}{2}d_i(\mathbf{x}^t)^T(\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i)d_i(\mathbf{x}^t) + h_{\mathbf{b}}(\mathbf{x}_i^t + d_i(\mathbf{x}^t))] \\
\leq\ & \mathbb{E}[f(\mathbf{x}^t) + \langle \nabla_i f(\mathbf{x}^t), d_i(\mathbf{x}^t)\rangle + \tfrac{L}{2}\|d_i(\mathbf{x}^t)\|_2^2 + h_{\mathbf{b}}(\mathbf{x}_i^t + d_i(\mathbf{x}^t))] \\
\leq\ & \mathbb{E}[f(\mathbf{x}^t) + \langle \nabla_i f(\mathbf{x}^t), d_i(\mathbf{x}^t)\rangle + \tfrac{L+\theta}{2}\|d_i(\mathbf{x}^t)\|_2^2 + h_{\mathbf{b}}(\mathbf{x}_i^t + d_i(\mathbf{x}^t))] \\
\leq\ & \mathbb{E}[f(\mathbf{x}^t) + \langle \nabla_i f(\mathbf{x}^t), d_i'(\mathbf{x}^t)\rangle + \tfrac{L+\theta}{2}\|d_i'(\mathbf{x}^t)\|_2^2 + h_{\mathbf{b}}(\mathbf{x}_i^t + d_i'(\mathbf{x}^t))] \\
=\ & f(\mathbf{x}^t) + \mathbb{E}[\langle \nabla_i f(\mathbf{x}^t), d_i'(\mathbf{x}^t)\rangle + \tfrac{L+\theta}{2}\|d_i'(\mathbf{x}^t)\|_2^2 + 0] \\
=\ & f(\mathbf{x}^t) + \tfrac{k}{n}\langle \nabla f(\mathbf{x}^t), d'(\mathbf{x}^t)\rangle + \tfrac{k}{n}\tfrac{L+\theta}{2}\|d'(\mathbf{x}^t)\|_2^2
\end{aligned} \tag{15}$$

where the first step uses the definition of $\mathbf{x}^{t+1}$; the second step uses the definition of $\mathbf{d}_i$; the third step uses the inequality that $\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i \leq L, \forall i$; the fourth step uses the fact that $d_i'(\mathbf{x}^k)$ is the minimizer of $H_i(\cdot)$ for all i; the fifth step uses the definition of the whole working set.

For any $\mathbf{x}^t \in \Psi$ and $\mathbf{x}^{t+1} \in \Psi$, we naturally derive the following inequalities:

$$\begin{aligned}
& \tfrac{n}{k}\mathbb{E}[f(\mathbf{x}^{t+1})|\mathbf{x}^t] - \tfrac{n}{k}f(\mathbf{x}^t) \\
\leq\ & \langle \nabla f(\mathbf{x}^t), d'(\mathbf{x}^t)\rangle + \tfrac{L+\theta}{2}d'(\mathbf{x}^t)^Td'(\mathbf{x}^t) \\
=\ & \langle \nabla f(\mathbf{x}^t), \Pi\left(\mathbf{x}^t - \nabla f(\mathbf{x}^t)/(L+\theta)\right) - \mathbf{x}^k\rangle + \tfrac{L+\theta}{2}\|\Pi\left(\mathbf{x}^t - \nabla f(\mathbf{x}^t)/(L+\theta)\right) - \mathbf{x}^t\|_2^2 \\
=\ & \tfrac{L+\theta}{2}\|\Pi\left(\mathbf{x}^t - \nabla f(\mathbf{x}^t)/(L+\theta)\right) - \mathbf{x}^t + \nabla f(\mathbf{x}^t)/(L+\theta)\|_2^2 - \tfrac{L+\theta}{2}\|\nabla f(\mathbf{x}^t)/(L+\theta)\|_2^2 \\
\leq\ & \tfrac{(L+\theta)(1-\kappa)}{2}\|\Pi(\bar{\mathbf{x}}) - \mathbf{x}^t + \nabla f(\mathbf{x}^t)/(L+\theta)\|_2^2 - \tfrac{L+\theta}{2}\|\nabla f(\mathbf{x}^t)/(L+\theta)\|_2^2 \\
=\ & \tfrac{(L+\theta)(1-\kappa)}{2}\|\bar{\mathbf{x}} - \mathbf{x}^t\|_2^2 + (1-\kappa)\langle \bar{\mathbf{x}} - \mathbf{x}^t, \nabla f(\mathbf{x}^t)\rangle - \tfrac{\kappa}{2}\|\nabla f(\mathbf{x}^t)/(L+\theta)\|_2^2 \\
\leq\ & \tfrac{(L+\theta-s)(1-\kappa)}{2}\|\bar{\mathbf{x}} - \mathbf{x}^t\|_2^2 - \tfrac{\kappa}{2(L+\theta)^2}\|\nabla f(\mathbf{x}^t)\|_2^2 + (1-\kappa)(f(\bar{\mathbf{x}}) - f(\mathbf{x}^t)) \\
=\ & (\tfrac{\kappa}{2(L+\theta)^2})\left(\tfrac{(L+\theta-s)(1-\kappa)(L+\theta)^2}{\kappa}\|\bar{\mathbf{x}} - \mathbf{x}^t\|_2^2 - \|\nabla f(\mathbf{x}^t)\|_2^2\right) + (1-\kappa)(f(\bar{\mathbf{x}}) - f(\mathbf{x}^t)) \\
\leq\ & (\tfrac{\kappa}{2(L+\theta)^2})\sqrt{\tfrac{(L+\theta-s)(1-\kappa)(L+\theta)^2}{\kappa}}(f(\bar{\mathbf{x}}) - f(\mathbf{x}^t)) + (1-\kappa)(f(\bar{\mathbf{x}}) - f(\mathbf{x}^t))
\end{aligned}$$

where the second step uses the fact that the problem of $\arg\min_{\mathbf{d}\in\mathbb{R}^n} H_i'(\mathbf{x},\mathbf{d})$ admits a closed-form solution with $d'(\mathbf{x}) = \Pi\left(\mathbf{x} - \nabla f(\mathbf{x})/(L+\theta)\right) - \mathbf{x}$; the third step uses the inequality in (12) in Lemma 2; the fourth step uses the fact that $\Pi(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$; the fifth step uses the strongly convexity of $f(\cdot)$ that $f(\mathbf{x}^t) - f(\bar{\mathbf{x}}) \leq \langle \nabla f(\mathbf{x}^t), \mathbf{x}^t - \mathbf{x}^*\rangle - \frac{s}{2}\|\mathbf{x}^t - \bar{\mathbf{x}}\|_2^2$; the seventh step uses the inequality in (13).

We have the following inequality: $E[f(\mathbf{x}^{k+1}) \mid \mathbf{x}^k] - f(\mathbf{x}^k) \leq C(f(\bar{\mathbf{x}}) - f(\mathbf{x}^t))$. Rearranging terms, we obtain the inequality in (14). In other words, the sequence $\{f(\mathbf{x}^t)\}$ converges to the stationary point linearly in the quotient sense. Applying the inequality in (14) recursively, we obtain: $E[f(\mathbf{x}^{k+1})] - f(\bar{\mathbf{x}}) \leq (1-C)^k(f(\mathbf{x}^0) - f(\bar{\mathbf{x}}))$. Therefore, we conclude that it takes at most $\log_{(1-C)}(\frac{\epsilon}{F(\mathbf{x}^0)-F(\bar{\mathbf{x}})})$ times to find a local optimal solution satisfying $F(\mathbf{x}^k) - F(\bar{\mathbf{x}}) \leq \epsilon$.

□

Remarks: We assume $f(\cdot)$ is strongly convex. However, this assumption always holds for binary optimization. This is because one can append an additional term $\frac{\eta}{2}\|\mathbf{x}\|_2^2$ to the objective function $f(\cdot)$ and the objective function becomes strongly convex with sufficiently large η.

A.2. Proof of Theorem 4 (Convergence Rate for Sparse Optimization)

In what follows, we establish the convergence rate for sparse optimization. We have derived the bound of the number of change T_{out} for the support set in Theorem 1. Now we need to derive a bound on the number of iterations T_{in} performed after the support set is fixed. Combing these two bounds, we complete our proof.

We notice that when the support set is fixed, the original problem reduces to a convex composite optimization problem. For notational convenience, we define:

$$\begin{aligned} J(\mathbf{x}) &\triangleq f(\mathbf{x}) + p(\mathbf{x}), \text{ with } p(\mathbf{x}) \triangleq I_{\Omega}(\mathbf{x}) \\ \mathbf{d}_i(\mathbf{x}) &\triangleq \arg\min_{\mathbf{d}_i \in \mathbb{R}^k} H_i(\mathbf{x}, \mathbf{d}_i) \triangleq f(\mathbf{x}) + \langle \nabla_i f(\mathbf{x}), \mathbf{d}_i \rangle + \tfrac{1}{2}\mathbf{d}_i^T(\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i + \theta\mathbf{I})\mathbf{d}_i + p_i(\mathbf{x}_i + \mathbf{d}_i) \\ \bar{\mathbf{d}}(\mathbf{x}) &\triangleq \tfrac{1}{C_n^k}\textstyle\sum_{i=1}^{C_n^k}\mathbf{U}_i\mathbf{d}_i(\mathbf{x}) \end{aligned} \tag{16}$$

Lemma 4. *For any $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$, if we pick an arbitrary combination $i \in \{1, 2, ..., C_n^k\}$ from the whole working set uniformly at random, then*

$$\tfrac{k}{n}\langle (\mathbf{Q} + \theta\mathbf{I})\bar{\mathbf{d}}, \mathbf{x} - \mathbf{y}\rangle \leq (1 - \tfrac{k}{n})J(\mathbf{x}) + \tfrac{k}{n}J(\mathbf{y}) - \mathbb{E}[F(\mathbf{x} + \mathbf{U}_i\bar{\mathbf{d}}_i)] + \tfrac{k}{2n}\bar{\mathbf{d}}(\mathbf{x})^T(L\mathbf{I} - 2\theta\mathbf{I} - 2\mathbf{Q})\bar{\mathbf{d}}(\mathbf{x})$$

Proof. We naturally derive the following inequalities:

$$\begin{aligned} & \mathbb{E}[F(\mathbf{x} + \mathbf{U}_i\mathbf{d}_i)] \\ \leq\ & \mathbb{E}[f(\mathbf{x}) + \langle \nabla f(\mathbf{x})_i, \mathbf{d}_i(\mathbf{x})\rangle + \tfrac{1}{2}\mathbf{d}_i(\mathbf{x})^T(\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i)\mathbf{d}_i(\mathbf{x}) + p_i(\mathbf{x}_i + \mathbf{d}_i)] \\ \leq\ & \mathbb{E}[f(\mathbf{x}) + \langle \nabla f(\mathbf{x})_i, \mathbf{d}_i(\mathbf{x})\rangle + \tfrac{L}{2}\|\mathbf{d}_i(\mathbf{x})\|_2^2 + p_i(\mathbf{x}_i + \mathbf{d}_i)] \\ =\ & f(\mathbf{x}) + \tfrac{k}{n}\langle \nabla f(\mathbf{x}), \bar{\mathbf{d}}(\mathbf{x})\rangle + \tfrac{L}{2}\tfrac{k}{n}\|\bar{\mathbf{d}}(\mathbf{x})\|_2^2 + \tfrac{k}{n}p(\mathbf{x} + \bar{\mathbf{d}}) + (1 - \tfrac{k}{n})p(\mathbf{x}) \\ =\ & f(\mathbf{x}) - \tfrac{k}{n}f(\mathbf{x}) + \tfrac{k}{n}f(\mathbf{x}) + \tfrac{k}{n}\langle \nabla f(\mathbf{x}), \bar{\mathbf{d}}(\mathbf{x})\rangle + \tfrac{L}{2}\tfrac{k}{n}\|\bar{\mathbf{d}}(\mathbf{x})\|_2^2 + \tfrac{k}{n}p(\mathbf{x} + \bar{\mathbf{d}}) + (1 - \tfrac{k}{n})p(\mathbf{x}) \\ =\ & (1 - \tfrac{k}{n})J(\mathbf{x}) + \tfrac{k}{n}f(\mathbf{x}) + \tfrac{k}{n}\langle \nabla f(\mathbf{x}), \bar{\mathbf{d}}(\mathbf{x})\rangle + \tfrac{L}{2}\tfrac{k}{n}\|\bar{\mathbf{d}}(\mathbf{x})\|_2^2 + \tfrac{k}{n}p(\mathbf{x} + \bar{\mathbf{d}}) \\ \leq\ & (1 - \tfrac{k}{n})J(\mathbf{x}) + \tfrac{k}{n}f(\mathbf{y}) + \tfrac{k}{n}\langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} + \bar{\mathbf{d}}(\mathbf{x})\rangle + \tfrac{L}{2}\tfrac{k}{n}\|\bar{\mathbf{d}}(\mathbf{x})\|_2^2 + \tfrac{k}{n}p(\mathbf{x} + \bar{\mathbf{d}}) \\ \leq\ & (1 - \tfrac{k}{n})J(\mathbf{x}) + \tfrac{k}{n}f(\mathbf{y}) + \tfrac{k}{n}\langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} + \bar{\mathbf{d}}(\mathbf{x})\rangle + \tfrac{L}{2}\tfrac{k}{n}\|\bar{\mathbf{d}}(\mathbf{x})\|_2^2 + \tfrac{k}{n}(p(\mathbf{y}) + \langle \partial p(\mathbf{x} + \bar{\mathbf{d}}), \mathbf{x} - \mathbf{y} + \bar{\mathbf{d}}\rangle) \\ \leq\ & (1 - \tfrac{k}{n})J(\mathbf{x}) + \tfrac{k}{n}f(\mathbf{y}) + \tfrac{k}{n}\langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} + \bar{\mathbf{d}}(\mathbf{x})\rangle + \tfrac{L}{2}\tfrac{k}{n}\|\bar{\mathbf{d}}(\mathbf{x})\|_2^2 + \tfrac{k}{n}(p(\mathbf{y}) - \langle \nabla f(\mathbf{x}) + (\mathbf{Q} + \theta\mathbf{I})\bar{\mathbf{d}}, \mathbf{x} - \mathbf{y} + \bar{\mathbf{d}}\rangle) \\ =\ & (1 - \tfrac{k}{n})J(\mathbf{x}) + \tfrac{k}{n}J(\mathbf{y}) + \tfrac{L}{2}\tfrac{k}{n}\|\bar{\mathbf{d}}(\mathbf{x})\|_2^2 - \tfrac{k}{n}\langle (\mathbf{Q} + \theta\mathbf{I})\bar{\mathbf{d}}, \mathbf{x} - \mathbf{y} + \bar{\mathbf{d}}\rangle \\ =\ & (1 - \tfrac{k}{n})J(\mathbf{x}) + \tfrac{k}{n}J(\mathbf{y}) + \tfrac{k}{2n}\bar{\mathbf{d}}(\mathbf{x})^T(L\mathbf{I} - 2\theta\mathbf{I} - 2\mathbf{Q})\bar{\mathbf{d}}(\mathbf{x}) - \tfrac{k}{n}\langle (\mathbf{Q} + \theta\mathbf{I})\bar{\mathbf{d}}, \mathbf{x} - \mathbf{y}\rangle \end{aligned}$$

where the first step uses the fact that $f(\mathbf{x} + \mathbf{U}_i\mathbf{d}_i) \leq f(\mathbf{x}) + \langle \nabla_i f(\mathbf{x}), \mathbf{d}_i\rangle + \frac{1}{2}\mathbf{d}_i^T\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i\mathbf{d}_i$ holds due to the convexity of $f(\cdot)$; the second step uses the inequality $\max_{i\in\Gamma}\|\mathbf{U}_i^T\mathbf{Q}\mathbf{U}_i\| \leq L$; the third step uses the fact that $\mathbb{E}[\mathbf{U}_i\mathbf{x}_i] = \frac{k}{n}\cdot\frac{1}{C_n^k}\cdot\sum_{i=1}^{C_n^k}\mathbf{U}_i\mathbf{x}_i,\ \forall \mathbf{x} \in \mathbb{R}^n$; the sixed step uses the convexity of $f(\cdot)$ that $f(\mathbf{x}) \leq f(\mathbf{y}) + \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y}\rangle,\ \forall \mathbf{y} \in \mathbb{R}^n$; the seventh step uses the convexity of $p(\cdot)$ that $p(\mathbf{z}) \leq p(\mathbf{y}) + \langle \partial p(\mathbf{z}), \mathbf{z} - \mathbf{y}\rangle,\ \forall \mathbf{y}, \mathbf{z}$; the eighth step uses the first-order optimality condition of the optimization in $\bar{\mathbf{d}}(\mathbf{x})$, we have: $-\nabla f(\mathbf{x}) - (L + \theta)\bar{\mathbf{d}}(\mathbf{x}) \in \partial p(\mathbf{x} + \bar{\mathbf{d}}(\mathbf{x}))$. Rearranging terms, we quickly finish the proof of this lemma. □

Proposition 2. *We denote $\kappa \triangleq (1 - \frac{k}{n})L + \frac{k}{n}\theta$. When the support set does not change, it takes T_{in} iterations for Algorithm 1 to converge to the local minima $\bar{\mathbf{x}}$ with*

$$T_{in} \leq \frac{n}{k} \times \frac{R(L + \theta) + (1 + \frac{\kappa}{\theta})(F(\mathbf{x}) - F(\bar{\mathbf{x}}))}{F(\mathbf{x}^t) - F(\bar{\mathbf{x}})}$$

Proof. When the support set does not change, the optimization problem reduced to the convex optimization as in (16). Let $\bar{\mathbf{x}}$ be an arbitrary optimal solution. We denote: $r_t^2 = \|\mathbf{x}^t - \mathbf{x}^*\|_{(\mathbf{A}+\theta\mathbf{I})}^2 = \langle \mathbf{x}^t - \bar{\mathbf{x}}, (\mathbf{A} + \theta\mathbf{I})(\mathbf{x}^t - \bar{\mathbf{x}})\rangle$. Noticing that

$\mathbf{x}^{t+1} = \mathbf{x}^t + \mathbf{U}_i d_i(\mathbf{x}_t)$, we have

$$\tfrac{1}{2}r_{t+1}^2 = \tfrac{1}{2}r_t^2 + \langle(\mathbf{Q}+\mathbf{I})\mathbf{U}_i d_i(\mathbf{x}^t), \mathbf{x}^t - \mathbf{x}^*\rangle + \|\mathbf{U}_i d_i(\mathbf{x}^t)\|_{(\mathbf{Q}+\theta\mathbf{I})}^2 \tag{17}$$

Taking expectation with respect to B, we obtain:

$$\begin{aligned}
& \mathbb{E}[\tfrac{1}{2}r_{t+1}^2] \\
= \quad & \tfrac{1}{2}r_t^2 + \frac{k}{2n}\|d(\mathbf{x}_k)\|_{(\mathbf{Q}+\theta\mathbf{I})}^2 + \frac{k}{n}\langle(\mathbf{Q}+\mathbf{I})d(\mathbf{x}^t), \mathbf{x}^t - \mathbf{x}^*\rangle \\
\leq \quad & \tfrac{1}{2}r_t^2 + \tfrac{k}{2n}\|d(\mathbf{x}^t)\|_{(\mathbf{Q}+\theta\mathbf{I})}^2 + (1-\tfrac{k}{n})J(\mathbf{x}^t) + \tfrac{k}{n}J(\mathbf{y}) - \mathbb{E}[J(\mathbf{x}^t + \mathbf{U}_i\bar{\mathbf{d}})] + \tfrac{1}{2}\bar{\mathbf{d}}(\mathbf{x})^T(\mathbf{Q} - \frac{2k}{n}\mathbf{Q} - \frac{2k\theta}{n}\mathbf{I})\bar{\mathbf{d}} \\
= \quad & \tfrac{1}{2}r_j^2 + (1-\tfrac{k}{n})J(\mathbf{x}^t) + \tfrac{k}{n}J(\mathbf{y}) - \mathbb{E}[J(\mathbf{x}^t + \mathbf{U}_i\bar{\mathbf{d}}_i)] + \tfrac{1}{2}\bar{\mathbf{d}}^T(\mathbf{Q} - \frac{2k}{n}\mathbf{Q} - \frac{2k\theta}{n}\mathbf{I} + \frac{k}{n}\mathbf{Q} + \frac{k}{n}\theta\mathbf{I})\bar{\mathbf{d}} \\
= \quad & \tfrac{1}{2}r_t^2 + (1-\tfrac{k}{n})J(\mathbf{x}^t) + \tfrac{k}{n}J(\mathbf{y}) - \mathbb{E}[J(\mathbf{x}^t + \mathbf{U}_i\bar{\mathbf{d}}_i)] + \tfrac{1}{2}\bar{\mathbf{d}}^T(\mathbf{Q} - \frac{k}{n}\mathbf{Q} - \frac{k\theta}{n}\mathbf{I})\bar{\mathbf{d}}
\end{aligned}$$

Letting $\mathbf{y} = \mathbf{x}^*$, we obtain:

$$\mathbb{E}[\tfrac{1}{2}r_{i+1}^2] \leq \tfrac{1}{2}r_i^2 + (1-\tfrac{k}{n})J(\mathbf{x}^i) + \tfrac{k}{n}J(\mathbf{x}^*) - \mathbb{E}[J(\mathbf{x}^{i+1})] + \tfrac{\kappa}{2}\|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2$$

Let $i = 0, 1, ..., t$, we obtain:

$$\begin{aligned}
\mathbb{E}[\tfrac{1}{2}r_1^2] &\leq \tfrac{1}{2}r_0^2 + (1-\tfrac{k}{n})J(\mathbf{x}^0) + \tfrac{k}{n}F(\mathbf{x}^*) - \mathbb{E}[J(\mathbf{x}^1)] + \tfrac{\kappa}{2}\|\bar{\mathbf{d}}(\mathbf{x}^0)\|_2^2 \\
\mathbb{E}[\tfrac{1}{2}r_2^2] &\leq \tfrac{1}{2}r_1^2 + (1-\tfrac{k}{n})J(\mathbf{x}^1) + \tfrac{k}{n}F(\mathbf{x}^*) - \mathbb{E}[J(\mathbf{x}^2)] + \tfrac{\kappa}{2}\|\bar{\mathbf{d}}(\mathbf{x}^1)\|_2^2 \\
&\dots \\
\mathbb{E}[\tfrac{1}{2}r_{t+1}^2] &\leq \tfrac{1}{2}r_j^2 + (1-\tfrac{k}{n})J(\mathbf{x}^t) + \tfrac{k}{n}F(\mathbf{x}^*) - \mathbb{E}[J(\mathbf{x}^{t+1})] + \tfrac{\kappa}{2}\|\bar{\mathbf{d}}(\mathbf{x}^t)\|_2^2
\end{aligned}$$

Therefore, we have the following inequalities:

$$\begin{aligned}
& \mathbb{E}[\tfrac{1}{2}r_{t+1}^2] \leq \tfrac{1}{2}r_0^2 + J(\mathbf{x}^0) - J(\mathbf{x}^t) + \tfrac{k}{n}\textstyle\sum_{i=0}^t J(\mathbf{x}^*) - J(\mathbf{x}^i) + \tfrac{\kappa}{2}\textstyle\sum_{i=0}^t \|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2 \\
\Rightarrow \quad & \mathbb{E}[\tfrac{1}{2}r_{t+1}^2] \leq \tfrac{1}{2}r_0^2 + J(\mathbf{x}^0) - J(\mathbf{x}^*) + \tfrac{k}{n}\textstyle\sum_{i=0}^t J(\mathbf{x}^*) - J(\mathbf{x}^i) + \tfrac{\kappa}{2}\textstyle\sum_{i=0}^t \|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2 \\
\Rightarrow \quad & \tfrac{k}{n}\textstyle\sum_{i=0}^t J(\mathbf{x}^i) - J(\mathbf{x}^*) \leq \tfrac{1}{2}r_0^2 - \mathbb{E}[\tfrac{1}{2}r_{t+1}^2] + F(\mathbf{x}^0) - J(\mathbf{x}^*) + \tfrac{\kappa}{2}\textstyle\sum_{i=0}^t \|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2 \\
\Rightarrow \quad & \textstyle\sum_{i=0}^t J(\mathbf{x}^i) - J(\mathbf{x}^*) \leq \tfrac{n}{k}\cdot(\tfrac{1}{2}r_0^2 - \mathbb{E}[\tfrac{1}{2}r_{t+1}^2] + J(\mathbf{x}^0) - J(\mathbf{x}^*) + \tfrac{\kappa}{2}\textstyle\sum_{i=0}^t \|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2) \\
\Rightarrow \quad & t(J(\mathbf{x}^t) - J(\mathbf{x}^*)) \leq \textstyle\sum_{i=0}^t J(\mathbf{x}^i) - J(\mathbf{x}^*) \leq \tfrac{n}{k}\cdot(\tfrac{1}{2}r_0^2 - \mathbb{E}[\tfrac{1}{2}r_{t+1}^2] + J(\mathbf{x}^0) - J(\mathbf{x}^*) + \tfrac{\kappa}{2}\textstyle\sum_{i=0}^t \|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2)
\end{aligned}$$

Therefore, we have:

$$t \leq \frac{n(\frac{1}{2}r_0^2 - \frac{1}{2}r_{t+1}^2 + J(\mathbf{x}^0) - J(\mathbf{x}^*) + \frac{\kappa}{2}\sum_{i=0}^t \|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2)}{k(J(\mathbf{x}^t) - J(\mathbf{x}^*))}$$

We notice that $\frac{1}{2}\sum_{i=0}^t \|\bar{\mathbf{d}}(\mathbf{x}^i)\|_2^2) \leq \frac{F(\mathbf{x}) - F(\bar{\mathbf{x}})}{\theta}$ by Theorem 1. Combing the fact that $F(\mathbf{x}^0) \geq J(\mathbf{x}^0)$, $F(\mathbf{x}^*) \leq J(\mathbf{x}^*)$, $\|\mathbf{x}^t - \mathbf{x}^*\|_{(\mathbf{A}+\theta\mathbf{I})} \leq \|\mathbf{x}^t - \mathbf{x}^*\|(L+\theta) \leq (L+\theta)2R$, we finish the proof of this proposition.

□

Theorem 3. ***Proof of Convergence Rate when*** $f(\cdot)$ ***is Convex and*** $h = f_{sparse}$**.** *Algorithm 1 at most takes $T_{in} \times T_{out}$ iterations to find a local optimal solution satisfying $F(\mathbf{x}^k) - F(\bar{\mathbf{x}}) \leq \epsilon$. Here $T_{in} \triangleq 2C_n^k(F(\mathbf{x}^0) - F(\bar{\mathbf{x}}))/(\theta\delta)$ with $\delta \triangleq \min_j \{\min(\rho, |\mathbf{x}_j^0|, \sqrt{2\lambda/(\theta + \mathbf{Q}_{j,j})})\}$ is the upper bound for the number of changes in expectation (refer to Theorem 1). T_{out} defined in (17) is the upper bound for the number of iterations for finding the local minimum solution when the support set does not change (refer to Proposition 2).*

Proof. From the conclusion in Proposition 2, we conclude that it takes T_{in} iterations in expectation to converge to a local optimal solution that satisfies $F(\mathbf{x}^k) - F(\mathbf{x}^*) \leq \epsilon$. Moreover, from Theorem 1 we have that Algorithm 1 changes at most T_{out}. Therefore, we conclude that it takes $T_{\text{in}} \times T_{\text{out}}$ iteration to converge to the local optimal solution. This finishes the proof of this Theorem.

□